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ON THE RANGE OF A COVERING FUNCTION

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ABSTRACT. Let $\{a_s \pmod{n_s}\}_{s=1}^k$ ($k > 1$) be a finite system of residue classes with the moduli n_1, \dots, n_k distinct. By means of algebraic integers we show that the range of the covering function $w(x) = |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|$ is not contained in any residue class with modulus greater one. In particular, the values of $w(x)$ cannot have the same parity.

1. INTRODUCTION

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, let $a(n)$ stand for the residue class $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. A finite system

$$\{a_s(n_s)\}_{s=1}^k \quad (k > 1) \tag{1.1}$$

of residue classes is said to be a *cover* of \mathbb{Z} if $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$.

The concept of cover of \mathbb{Z} was introduced by P. Erdős ([E50]) in the early 1930s, who was particularly interested in those covers (1.1) with the moduli n_1, \dots, n_k distinct. By Example 3 of the author [S96], if $n > 1$ is odd then

$$\{1(2), 2(2^2), \dots, 2^{n-2}(2^{n-1}), 2^{n-1}(n), 2^{n-1}2(2n), \dots, 2^{n-1}n(2^{n-1}n)\}$$

forms a cover of \mathbb{Z} with distinct moduli. Covers of \mathbb{Z} have been studied by various researchers (cf. [G04] and [PJ]) and many surprising applications have been found (see, e.g. [F02], [S00], [S01] and [S03b]).

Here are two major open problems concerning covers of \mathbb{Z} (see sections E23, F13 and F14 of [G04] for references to these and other conjectures).

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Erdős–Selfridge Conjecture. *Let (1.1) be a cover of \mathbb{Z} with distinct moduli. Then n_1, \dots, n_k cannot be all odd.*

Schinzel’s Conjecture. *If (1.1) is a cover of \mathbb{Z} , then there is a modulus n_t dividing another modulus n_s .*

For system (1.1), the function $w : \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$w(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}| \quad (1.2)$$

is called its *covering function*. Obviously $w(x)$ is periodic modulo the least common multiple $N = [n_1, \dots, n_k]$ of the moduli n_1, \dots, n_k .

Now we list some known results concerning the covering function $w(x)$.

- (i) The arithmetic mean of $w(x)$ with x in a period equals $\sum_{s=1}^k 1/n_s$.
- (ii) (Z. W. Sun [S95, S96]) The covering function $w(x)$ takes its minimum on every set of

$$\left| \left\{ 0 \leq \theta < 1 : \sum_{s \in I} \frac{m_s}{n_s} - \theta \in \mathbb{Z} \text{ for some } I \subseteq \{1, \dots, k\} \right\} \right|$$

consecutive integers, where m_1, \dots, m_k are given integers relatively prime to n_1, \dots, n_k respectively.

- (iii) (Z. W. Sun [S03a]) The maximal value of $w(x)$ can be written in the form $\sum_{s=1}^k m_s/n_s$ with $m_1, \dots, m_k \in \mathbb{Z}^+$.

- (iv) (Š. Porubský [P75]) If n_1, \dots, n_k are distinct, then $[n_1, \dots, n_k]$ is the smallest positive period of the function $w(x)$.

- (v) (Z. W. Sun [S03a]) If $n_0 \in \mathbb{Z}^+$ is a period of the function $w(x)$, then for any $t = 1, \dots, k$ we have

$$\left\{ \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \dots, k\} \setminus \{t\} \right\} \supseteq \left\{ \frac{r}{n_t} : r \in \mathbb{Z} \text{ and } 0 \leq r < \frac{n_t}{(n_0, n_t)} \right\},$$

where (n_0, n_t) denotes the greatest common divisor of n_0 and n_t .

- (vi) (Z. W. Sun [S04]) The function $w(x)$ is constant if $w(x)$ equals a constant for $|S|$ consecutive integers x where

$$S = \left\{ \frac{r}{n_s} : r = 0, \dots, n_s - 1; s = 1, \dots, k \right\}.$$

In this paper we study the range of a covering function (via algebraic integers) for the first time. Proofs of the theorems below will be given in the next section.

Theorem 1.1. *Suppose that the range of the covering function of (1.1) is contained in a residue class with modulus m . Then, for any $t = 1, \dots, k$ with $mn_t \nmid [n_1, \dots, n_k]$, we have $n_t \mid n_s$ for some $1 \leq s \leq k$ with $s \neq t$.*

Corollary 1.1. *If the covering function $w(x)$ of (1.1) is constant, then for any $t = 1, \dots, k$ there is an $s \neq t$ such that $n_t \mid n_s$, and in particular $n_k = n_{k-1}$ provided that $n_1 \leq \dots \leq n_{k-1} \leq n_k$.*

Proof. Suppose that $w(x) = c$ for all $x \in \mathbb{Z}$. Choose an integer $m > [n_1, \dots, n_k]$. As $c(m)$ contains the range of $w(x)$, the desired result follows from Theorem 1.1. \square

Remark 1.1. When (1.1) is a disjoint cover of \mathbb{Z} , i.e., $w(x) = 1$ for all $x \in \mathbb{Z}$, the first part of Corollary 1.1 was given by B. Novák and Š. Znáám [NZ] and the second part was originally obtained by H. Davenport, L. Mirsky, D. Newman and R. Radó independently. Corollary 1.1 appeared in Porubský [P75].

Corollary 1.2. *Suppose that those moduli in (1.1) which are maximal with respect to divisibility are distinct. Then $w(\mathbb{Z}) = \{w(x) : x \in \mathbb{Z}\}$ cannot be contained in a residue class other than $0(1) = \mathbb{Z}$, i.e., for any prime p there is an $x \in \mathbb{Z}$ with $w(x) \not\equiv w(0) \pmod{p}$. In particular, those $w(x)$ with $x \in \mathbb{Z}$ cannot have the same parity.*

Proof. Assume that $w(\mathbb{Z})$ is contained in a residue class with modulus $m \in \mathbb{Z}^+$. For each modulus n_t maximal with respect to divisibility, there is no $s \neq t$ such that $n_t \mid n_s$, thus mn_t divides $N = [n_1, \dots, n_k]$ by Theorem 1.1. Since N is also the least common multiple of those moduli n_t maximal with respect to divisibility, we must have $mN \mid N$ and hence $m = 1$. This ends the proof. \square

Remark 1.2. In contrast with the Erdős–Selfridge conjecture, Corollary 1.2 indicates that **if (1.1) is a cover of \mathbb{Z} with distinct moduli then not every integer is covered by (1.1) odd times.**

Here is another related result.

Theorem 1.2. *Let $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ both have distinct moduli. Then A and B are identical provided that $w_A(x) \equiv w_B(x) \pmod{m}$ for all $x \in \mathbb{Z}$, where w_A and w_B are covering functions of A and B respectively, and m is an integer not dividing $N = [n_1, \dots, n_k, m_1, \dots, m_l]$.*

Remark 1.3. In 1975 Znáám [Z75] extended a uniqueness theorem of S. K. Stein [St] as follows: Under the condition of Theorem 1.2, we have $A = B$ if $w_A = w_B$. This follows from Theorem 1.2 by taking $m > N$.

Theorem 1.1 can be refined as follows.

Theorem 1.3. *Let $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$ be weights assigned to the k residue classes in (1.1) respectively. Suppose that $n_0 \in \mathbb{Z}^+$ is the smallest positive period of $w(x) = \sum_{1 \leq s \leq k, n_s \mid x - a_s} \lambda_s$ modulo $m \in \mathbb{Z}$, and that $d \in \mathbb{Z}^+$ does*

not divide n_0 but $I(d) = \{1 \leq s \leq k : d \mid n_s\} \neq \emptyset$. Then, either m divides $[n_1, \dots, n_k] \sum_{s \in I(d)} \lambda_s / n_s$, or we have

$$|I(d)| \geq |\{a_s \bmod d : s \in I(d)\}| \geq \min_{\substack{0 \leq s \leq k \\ s \notin I(d)}} \frac{d}{(d, n_s)} \geq p(d) \quad (1.3)$$

where $p(d)$ denotes the smallest prime divisor of d .

Remark 1.4. Theorem 1.3 in the case $m = 0$ was first obtained by the author [S91] in 1991, an extension of this was given in [S04].

Instead of (1.1) we can also consider a finite system of residue classes in \mathbb{Z}^n (cf. [S04]) and deduce n -dimensional versions of Theorems 1.1–1.3.

2. PROOFS OF THEOREMS 1.1–1.3

Proof of Theorem 1.1. Without any loss of generality we assume that $0 \leq a_s < n_s$ for $s = 1, \dots, k$. Set $N = [n_1, \dots, n_k]$. Then

$$\begin{aligned} \sum_{r=0}^{N-1} w(r) z^r &= \sum_{r=0}^{N-1} \sum_{\substack{1 \leq s \leq k \\ n_s \mid a_s - r}} z^r = \sum_{s=1}^k \sum_{\substack{0 \leq r < N \\ r \in a_s(n_s)}} z^r \\ &= \sum_{s=1}^k z^{a_s} \sum_{0 \leq q < N/n_s} (z^{n_s})^q \\ &= \sum_{\substack{1 \leq s \leq k \\ z^{n_s} = 1}} \frac{N}{n_s} z^{a_s} + (1 - z^N) \sum_{\substack{1 \leq s \leq k \\ z^{n_s} \neq 1}} \frac{z^{a_s}}{1 - z^{n_s}}. \end{aligned}$$

Suppose that $w(r) = a + mq_r$ for each $r \in \mathbb{Z}$ where $a, q_r \in \mathbb{Z}$. If $\alpha \notin \mathbb{Z}$ but $\alpha N \in \mathbb{Z}$, then

$$\sum_{r=0}^{N-1} w(r) e^{2\pi i \alpha r} = m \sum_{r=0}^{N-1} q_r e^{2\pi i \alpha r}$$

and also

$$\sum_{r=0}^{N-1} w(r) e^{2\pi i \alpha r} = \sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{N}{n_s} e^{2\pi i \alpha a_s},$$

therefore we have the following congruence

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{N}{n_s} e^{2\pi i \alpha a_s} \equiv 0 \pmod{m} \quad (2.1)$$

in the ring of all algebraic integers.

If $1 \leq t \leq k$ and $n_t \mid n_s$ for no $s \in \{1, \dots, k\} \setminus \{t\}$, then by applying (2.1) with $\alpha = 1/n_t < 1$ we obtain that

$$\frac{N}{n_t} e^{2\pi i a_t / n_t} \equiv 0 \pmod{m}$$

and hence m divides N/n_t in \mathbb{Z} .

The proof of Theorem 1.1 is now complete. \square

Proof of Theorem 1.2. Without any loss of generality, we assume that $n_1 > \dots > n_k$ and $m_1 > \dots > m_l$. As $w_A(x) - w_B(x) \equiv 0 \pmod{m}$ for all $x \in \mathbb{Z}$, by modifying the proof of Theorem 1.1 slightly, we find that if $\alpha \notin \mathbb{Z}$ but $\alpha N \in \mathbb{Z}$ then

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^k \frac{N}{n_s} e^{2\pi i \alpha a_s} - \sum_{\substack{t=1 \\ \alpha m_t \in \mathbb{Z}}}^l \frac{N}{m_t} e^{2\pi i \alpha b_t} \equiv 0 \pmod{m}. \quad (2.2)$$

In the case $d = \max\{m_1, n_1\} > 1$, by applying (2.2) with $\alpha = 1/d$ and the hypothesis $m \nmid N$, we get that $m_1 = n_1$ and

$$\frac{N}{d} \left(e^{2\pi i a_1 / d} - e^{2\pi i b_1 / d} \right) \equiv 0 \pmod{m}.$$

If $a_1 \not\equiv b_1 \pmod{d}$, then $z = 1 - e^{2\pi i (b_1 - a_1) / d}$ is a zero of the monic polynomial $(-1)^{d-1} P(1-x) \in \mathbb{Z}[x]$ where $P(x) = (1-x^d)/(1-x) = 1+x+\dots+x^{d-1}$, hence z divides the constant term $P(1) = d$ of $P(1-x)$ in the ring of algebraic integers. As m does not divide N , we must have $a_1 \equiv b_1 \pmod{d}$ and so $a_1(n_1) = b_1(m_1)$. Now that

$$|\{1 < s \leq k : x \in a_s(n_s)\}| \equiv |\{1 < t \leq l : x \in b_t(m_t)\}| \pmod{m},$$

we can continue the above procedure to obtain that

$$a_2(n_2) = b_2(m_2), \dots, a_{\min\{k,l\}}(n_{\min\{k,l\}}) = b_{\min\{k,l\}}(m_{\min\{k,l\}}).$$

If $k \neq l$, say $k > l$, then $m\mathbb{Z}$ contains the range of the covering function of $\{a_s(n_s)\}_{s=l+1}^k$ and this contradicts Theorem 1.1 since $m \nmid [n_{l+1}, \dots, n_k]$ and $n_{l+1} > \dots > n_k$. So $A = B$ and we are done. \square

Proof of Theorem 1.3. Let $N = [n_1, \dots, n_k]$. Clearly $(n_0, N) \in n_0\mathbb{Z} + N\mathbb{Z}$ is also a period of $w(x) \pmod{m}$, so $(n_0, N) = n_0$ and hence $n_0 \mid N$. Observe that

$$\sum_{\substack{s=1 \\ x \in a_s(n_s)}}^k \lambda_s - \sum_{\substack{r=0 \\ x \in r(n_0)}}^{n_0-1} w(r) \equiv 0 \pmod{m}$$

for each $x \in \mathbb{Z}$. As in the proof of Theorem 1.1, if $c \in \mathbb{Z}$ and $d \nmid c$ then

$$\sum_{\substack{s=1 \\ (c/d)n_s \in \mathbb{Z}}}^k \lambda_s \frac{N}{n_s} e^{2\pi i \frac{c}{d} a_s} - \sum_{\substack{r=0 \\ (c/d)n_0 \in \mathbb{Z}}}^{n_0-1} w(r) \frac{N}{n_0} e^{2\pi i \frac{c}{d} r} \equiv 0 \pmod{m}. \quad (2.3)$$

For any $c \in \mathbb{Z}^+$ divisible by none of those $d/(d, n_s)$ with $0 \leq s \leq k$ and $s \notin I(d)$, we have

$$d \mid cn_s \iff \frac{d}{(d, n_s)} \mid c \iff d \mid n_s \iff s \in I(d),$$

therefore (2.3) yields that

$$\sum_{s \in I(d)} \lambda_s \frac{N}{n_s} e^{2\pi i \frac{c}{d} a_s} \equiv 0 \pmod{m}.$$

Let

$$R = \{0 \leq r < d : a_s \equiv r \pmod{d} \text{ for some } s \in I(d)\}$$

and suppose that $|R| < \min_{0 \leq s \leq k, s \notin I(d)} d/(d, n_s)$. By the above,

$$u_n := \sum_{r \in R} c_r \left(e^{2\pi i \frac{r}{d}} \right)^n \equiv 0 \pmod{m} \text{ for every } n = 1, \dots, |R|,$$

where $c_r = N \sum_{s \in I(d), a_s \equiv r \pmod{d}} \lambda_s / n_s \in \mathbb{Z}$. As $\{u_n\}_{n \geq 0}$ is a linear recurrence of order $|R|$ with characteristic polynomial $\prod_{r \in R} (x - e^{2\pi i r/d})$ whose coefficients are algebraic integers, we have $u_n \equiv 0 \pmod{m}$ for every $n = |R| + 1, |R| + 2, \dots$. In particular, $\sum_{r \in R} c_r = u_d \equiv 0 \pmod{m}$, i.e., m divides $N \sum_{s \in I(d)} \lambda_s / n_s$. We are done. \square

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